

Wednesday, Nov. 12, Lecture 31: Convolution

Last time... Fourier Transform

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad \text{Fourier Transform } \mathcal{F}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \quad \text{Inverse Fourier Transform } \mathcal{F}^{-1}$$

• Parseval: $\mathcal{F}: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is unitary, so

$$\langle f, g \rangle = \langle \mathcal{F}f, \mathcal{F}g \rangle = \langle \hat{f}, \hat{g} \rangle.$$

• Diagonalizes derivatives:

$$\widehat{-i \frac{d}{dx} f} = k \hat{f}$$

We now introduce an important idea.

Def: Convolution

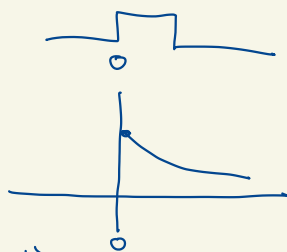
Let $f, g: \mathbb{R} \rightarrow \mathbb{C}$. The convolution $f * g: \mathbb{R} \rightarrow \mathbb{C}$ is defined by

$$\begin{aligned} (f * g)(x) &= \int_{-\infty}^{\infty} \underbrace{f(x-y)}_{\substack{\tau_x(f)(y) = f(y-x), \\ \text{then flip} \\ f(-(y-x)) = f(x-y)}} g(y) dy \\ &= (g * f)(x) \\ &= \int_{-\infty}^{\infty} g(x-y) f(y) dy \end{aligned}$$

Ex: Compute the convolution of

$$f(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

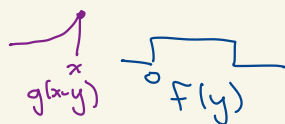


"Flip, translate by x , compute overlap".

3 regions:

• If $x < 0$

$$\begin{aligned} (g * f)(x) &= \int_{-\infty}^{\infty} f(y) g(x-y) dy \\ &= 0, \text{ no overlap.} \end{aligned}$$



• If $0 \leq x < 1$,

$$\begin{aligned} (g * f)(x) &= \int_{-\infty}^{\infty} f(y) g(x-y) dy \\ &= \int_0^x 1 \cdot e^{-2(x-y)} dy \\ &= \frac{1}{2} (1 - e^{-2x}). \end{aligned}$$



· If $x \geq 1$

$$(g * f)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

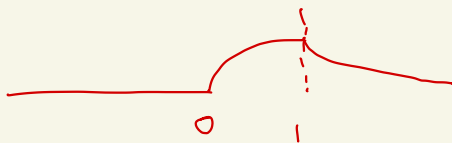
$$= \int_0^1 e^{-2(x-y)} dy$$

$$= e^{-2x} \left(\frac{1}{2}(1-e^{-2}) \right)$$



So:

$$(g * f)(x) =$$



Properties:

$$a) f * g = g * f$$

$$b) f * (g * h) = (f * g) * h$$

$$c) \tau_{\theta}(f * g) = (\tau_{\theta}f) * g = f * (\tau_{\theta}g)$$

$$\leftarrow (\tau_{\theta}f)(x) = f(x-\theta)$$

pf of c)

$$\tau_{\theta}(f * g)(x) = (f * g)(x-\theta)$$

$$= \int_{-\infty}^{\infty} f(x-\theta-y)g(y)dy$$

$$= \int_{-\infty}^{\infty} f(x-(y+\theta))g(y)dy$$

$$\begin{aligned} \tilde{y} &= y+\theta \\ d\tilde{y} &= dy \end{aligned}$$

$$= \int_{-\infty}^{\infty} f(x-\tilde{y})g(\tilde{y}-\theta)d\tilde{y}$$

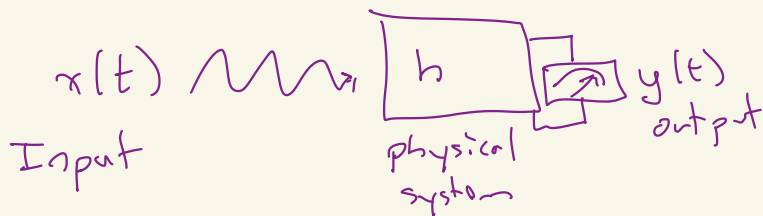
$$= (f * \tau_{\theta}g)(x).$$

The other one follows by $f * g = g * f$.



Turns out, many physical systems

(circuits, spring-mass-damper systems) can be modeled
as "Linear time invariant systems"



$$y(t) = (h * x)(t).$$

Proposition (Fourier Transform turns convolution into multiplication)

Let $f, g: \mathbb{R} \rightarrow \mathbb{C}$. Then

$$\widehat{f * g} = \sqrt{2\pi} \hat{f} \hat{g}.$$

Similarly for
inverse Fourier transform:
 $\check{f * g} = \sqrt{2\pi} \check{f} \check{g}$

pf:

$$\widehat{f * g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y) g(y) dy \right) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y) e^{-ik(x-y)} dx g(y) e^{-iky} dy$$

$$= \hat{f}(k) \int_{-\infty}^{\infty} g(y) e^{-iky} dy$$

$$= \sqrt{2\pi} \hat{f}(k) \hat{g}(k)$$



So, linear time invariant systems are often easier to understand in Fourier Domain.

Def: Dirac Delta "Function" ← Technically, a distribution, whatever that means.

Let $x \in \mathbb{R}$. We define the object δ_x by this property:
for all smooth $f: \mathbb{R} \rightarrow \mathbb{C}$,

$$\int_{-\infty}^{\infty} \delta_x \cdot f(y) dy = f(x)$$

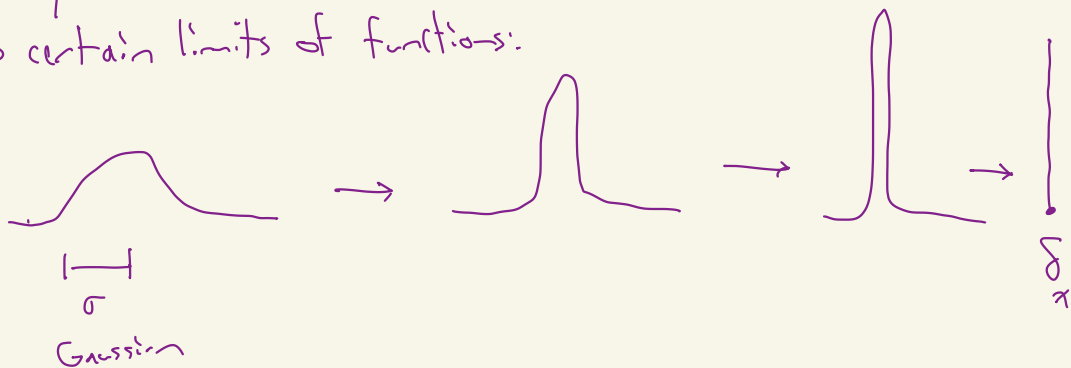
Physically, δ_x represent points with unit "size":

$$\int_{-\infty}^{\infty} \delta_x dy = 1$$

or, "unit impulse"

↓
 x

They are not functions, but we can think of them as certain limits of functions:



- This is how we can model point charges (like electrons), point masses (like distant stars), or "impulses" (like hitting a system with prescribed force for short time)

These are quite natural for several reasons.

Prop:

Let $f: \mathbb{R} \rightarrow \mathbb{C}$, $h \in \mathbb{R}$. Then

$$(\delta_h * f)(x) = f(x-h)$$

pf:

$$\begin{aligned}\delta_h * f(x) &= \int_{-\infty}^{\infty} \delta_h(y) f(x-y) dy \\ &= f(x-h)\end{aligned}$$



← Hey, it's the translation operator!

$$\tau_h = \delta_h * (\cdot)$$

Not: cc that if $h=0$, $\delta * f = f$.